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COMPLETE CLASS RESULTS FOR HYPOTHESIS TESTING PROBLEMS WITH SIMPLE NULL HYPOTHESES

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Hypothesis testing problems in which the null hypothesis is simple, the parameter space is finite dimensional and the supports of the probability measures are independent of the parameter are considered. Essentially complete class results are obtained for characterizing the limits of Bayes tests. Conditions for tests to be admissible and the class to be complete are given. Results are then specialized to exponential families, along with some illustrative examples.

1. Introduction. We are interested in characterizing admissibility for hypothesis testing problems in which there is a simple null hypothesis and a finite dimensional parameter space, and the supports of the probability distributions do not depend on the parameter. Cohen and Marden (1988, 1989) apply these results to invariance reduced problems. The 1988 paper considers the multivariate normal problems of testing for sphericity of a covariance matrix and testing the equality of two covariance matrices. The 1989 paper deals with Bartlett's problem of testing the equality of several univariate normal variances. Applications can also be made to problems of testing other structural hypotheses on covariance matrices and to combining independent two-sided tests. In Section 6 we present three examples: testing that the correlation is 0 in a bivariate normal distribution when the means and variances are known, testing that the location parameter of the shifted double exponential distribution is 0 and testing that a bivariate normal distribution has zero means versus the alternative that one mean is at least as large as the other in absolute value.

We assume a sample space $\mathscr{X}$, a parameter space $\Theta \subseteq \mathbb{R}^p$ which contains 0 and a family of distributions $\{P_\theta\colon \theta \in \Theta\}$ on a σ-field of $\mathscr{X}$. We wish to test

$$H_0\colon \theta = 0 \quad \text{versus} \quad H_A\colon \theta \in \Theta - \{0\}. \tag{1.1}$$

We assume that $P_\theta \ll P_{\theta'}$ for all $\theta, \theta' \in \Theta$. Hence we can let

$$f_\theta(x) = dP_\theta/dP_0 \tag{1.2}$$

be the density of P_θ with respect to the null measure P_0, which we will subsequently denote by ν. Most of our results apply to fairly general densities f_θ, but in Section 5 we specialize to the exponential case. That is, $\mathscr{X} \subseteq \mathbb{R}^p$ and

$$f_\theta(x) = e^{\theta^t x - \psi(\theta)}. \tag{1.3}$$

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The parameter space Θ must then be contained in the natural parameter space

$$\mathscr{N} \equiv \left\{\theta \in \mathbb{R}^p \middle| \int e^{\theta^t x} \nu(dx) < \infty\right\}. \tag{1.4}$$

Birnbaum (1955), Matthes and Truax (1967), Farrell (1968), Eaton (1970), Ghia (1976) and Marden (1982) present complete class results in special cases of the problem (1.1). The second and fourth papers also deal with problems involving nuisance parameters. The restrictions these papers place on their problems, however, are enough to rule out the problems mentioned in the first paragraph above; e.g., they restrict to exponential families or demand that 0 and $\Theta - \{0\}$ be topologically separated or require Θ to be contained in a pointed cone. See the introduction of Marden (1982) for a synopsis of these results. Kudô (1961) presents an approach to cases in which 0 and $\Theta - \{0\}$ are not topologically separated, but does not explicitly characterize the complete class. He also considers what he calls locally complete classes.

In Section 2 we present assumptions and prove a general essentially complete class theorem. This class contains extensions of the truncated generalized Bayes tests of Farrell (1968) and Ghia (1976). In Section 3 we give some sufficient conditions for a test to be admissible and for the class in Section 2 to be complete. Section 4 contains a description of the local terms in the essentially complete class. We specialize to the exponential family in Section 5 and in Section 6 we present some illustrative examples.

Throughout, we represent tests by measurable functions $\phi\colon \mathscr{X} \to [0,1]$ and use the risk function

$$r_\theta(\phi) = \begin{cases} E_\theta\phi, & \text{if } \theta = 0, \\ 1 - E_\theta\phi, & \text{if } \theta \in \Theta - \{0\}. \end{cases} \tag{1.5}$$

2. The essentially complete class. Our approach is to characterize the set of proper Bayes tests and their weak* limits [see (2.27)], which, under certain assumptions, yields an essentially complete class. See Wald (1950). When the alternative hypothesis space is compact, the set of proper Bayes tests is itself essentially complete. Problems arise when the alternative space is unbounded, in which case the sequence of prior probability measures corresponding to a convergent sequence of tests may place mass arbitrarily far from the null, and when the closure of the alternative space intersects the null, in which case the priors may concentrate mass arbitrarily close to the null. Assumption 2.3 is used to deal with the former problem and Assumption 2.2 and the sets defined in (2.18) to deal with the latter. We proceed with the definitions and assumptions which lead to the essentially complete class in Theorem 2.4.

ASSUMPTION 2.1 (Basic). There exists a positive function $a(\theta)$ with $a(0) = 1$ such that for each $x \in \mathscr{X}$,

$$R_\theta(x) \equiv a(\theta) f_\theta(x) \tag{2.1}$$

can be extended to a function on $\overline{\Theta}$ (the closure of Θ in $\mathbb{R}^p$) which is continuous for $\theta \in \overline{\Theta}$ and satisfies

$$0 < R_\theta(x) < \infty \quad \text{for all } \theta \in \overline{\Theta}. \tag{2.2}$$

ASSUMPTION 2.2 (Local). For each $x \in \mathscr{X}$, $R_\theta(x)$ has all first and second partial derivatives with respect to θ at $\theta = 0$. We denote these by

$$l(x) = \{l_i(x)\}_{i=1}^p, \qquad l_i(x) = \frac{\partial}{\partial\theta_i} R_\theta(x)\bigg|_{\theta=0} \tag{2.3}$$

and

$$V(x) = \{V_{ij}(x)\}_{i,j=1}^p, \qquad V_{ij}(x) = \frac{\partial^2}{\partial\theta_i\,\partial\theta_j} R_\theta(x)\bigg|_{\theta=0}. \tag{2.4}$$

Let $R_\theta^{(i)}$, $i = 1, 2$, be the following remainders from Taylor's expansion around $\theta = 0$:

$$\begin{aligned} R_\theta^{(1)}(x) &= R_\theta(x) - 1 - \theta^t l(x); \\ R_\theta^{(2)}(x) &= R_\theta(x) - 1 - \theta^t l(x) - \tfrac{1}{2}\langle \theta\theta^t, V(x)\rangle, \end{aligned} \tag{2.5}$$

where $\langle A, B\rangle = \operatorname{tr} AB$ if A and B are $p \times p$ matrices. For positive numbers $r < s$, define

$$\begin{aligned} \Theta(r) = \{\theta \in \Theta \mid 0 < \|\theta\| \le r\}, \qquad \Theta(r,s) = \{\theta \in \Theta \mid r < \|\theta\| \le s\}, \\ \Theta'(s) = \{\theta \in \Theta \mid \|\theta\| \ge s\}. \end{aligned} \tag{2.6}$$

Assume there exists a constant $\alpha > 0$ such that for each $x \in \mathscr{X}$,

$$\sup_{\theta \in \Theta(\alpha)} \left| \frac{R_\theta^{(2)}(x)}{\|\theta\|^2} \right| < \infty. \tag{2.7}$$

If there does exist a function $a(\theta)$ which satisfies Assumptions 2.1 and 2.2, then for any $x_0 \in \mathscr{X}$, the assumptions will hold for $a(\theta) = (f_\theta(x_0))^{-1}$. Also, if Assumption 2.2 holds for some $\alpha > 0$, it holds for all $\alpha > 0$.

For a Borel set $\Omega \subseteq \mathbb{R}^p$, let $\mathscr{P}(\Omega)$, $\mathscr{F}(\Omega)$ and $\mathscr{L}f(\Omega)$ denote, respectively, the set of probability, finite and locally finite Borel measures on Ω.

ASSUMPTION 2.3 (Asymptotic). There exists a collection $\mathscr{C}$ of closed subsets of $\mathscr{X}$ such that for any sequence $\{G_i\} \subseteq \mathscr{F}(\Theta)$, there exist a set $C \in \mathscr{C}$, a subsequence $\{i'\} \subseteq \{i\}$ and $G \in \mathscr{L}f(\overline{\Theta})$ such that

$$G_{i'} \to G \quad \text{vaguely}, \tag{2.8}$$

that is,

$$\int g(\theta) G_{i'}(d\theta) \to \int g(\theta) G(d\theta) \tag{2.9}$$

for all bounded continuous functions g with compact support and

$$(2.10)\qquad \int_\Theta R_\theta(x) G_{i'}(d\theta) \to \int_\Theta R_\theta(x) G(d\theta) < \infty \quad \text{for } x \in \operatorname{int} C$$

and

$$(2.11)\qquad \int_\Theta R_\theta(x) G_{i'}(d\theta) \to \infty \quad \text{for } x \in C^c.$$

We are implicitly assuming a topology on $\mathscr{X}$ and assume that the sets C are ν-measurable. Note that $\mathscr{C}$ always contains $\mathscr{X}$ and $\varnothing$. In exponential families, these sets C are convex and satisfy certain monotonicity conditions dictated by the structure of the alternative hypothesis space at alternatives far from the null. See Birnbaum (1955), Matthes and Truax (1967), Farrell (1968) and Eaton (1970). We treat this case in Section 5. Ghia (1976) presents similar results for more general densities. See also the "almost exponential" case in Marden (1982).

The quantities $\Lambda(H)$ and Δ_0 defined next reflect the structure of the alternative space near 0. See Section 4 for more details about these sets. For each $\varepsilon > 0$, let

$$(2.12)\quad \Delta_\varepsilon = \left\{ \int_{\Theta(\varepsilon)} \frac{(\theta, \theta\theta^t)}{\|\theta\|^2} H(d\theta) \,\middle|\, H \in \mathscr{F}(\Theta) \text{ and } \int_{\Theta(\varepsilon)} \frac{1}{\|\theta\|^2} H(d\theta) < \infty \right\},$$

where

$$\int_{\Theta(\varepsilon)} \frac{(\theta, \theta\theta')}{\|\theta\|^2} H(d\theta) = \left(\int_{\Theta(\varepsilon)} \frac{\theta}{\|\theta\|^2} H(d\theta), \int_{\Theta(\varepsilon)} \frac{\theta\theta'}{\|\theta\|^2} H(d\theta) \right).$$

Thus Δ_ε is a convex cone in $\mathbb{R}^p \times \mathscr{S}_p$, where $\mathscr{S}_p$ is the set of all $p \times p$ nonnegative definite matrices. For a given $H \in \mathscr{F}(\overline{\Theta}(\alpha) - \{0\})$, $(\lambda, M) \in \mathbb{R}^p \times \mathscr{S}_p$ and $\varepsilon \in (0, \alpha]$, set

$$(2.13)\quad (\lambda_\varepsilon, M_\varepsilon) \equiv (\lambda_\varepsilon(H), M_\varepsilon(H)) = (\lambda, M) - \int_{\overline{\Theta}(\varepsilon, \alpha)} \frac{(\theta, \theta\theta^t)}{\|\theta\|^2} H(d\theta).$$

Define $\Lambda(H) \subseteq \mathbb{R}^p \times \mathscr{S}_p$ by

$$(2.14)\qquad \Lambda(H) = \{(\lambda, M) | (\lambda_\varepsilon, M_\varepsilon) \in \overline{\Delta}_\varepsilon \text{ for every } \varepsilon \in (0, \alpha]\}.$$

Since $\theta\theta^t/\|\theta\|^2$ is bounded and continuous for $\theta \neq 0$, for given $(\lambda, M) \in \Lambda(H)$ we can set

$$(2.15)\qquad M_0 = M - \int_{\overline{\Theta}(\alpha) - \{0\}} \frac{\theta\theta^t}{\|\theta\|^2} H(d\theta) = \lim_{\varepsilon \downarrow 0} M_\varepsilon.$$

Similarly, if we have that

$$(2.16)\qquad \int_{\overline{\Theta}(\alpha) - \{0\}} \frac{1}{\|\theta\|} H(d\theta) < \infty,$$

then we can define

$$\lambda_0 = \lambda - \int_{\overline{\Theta}(\alpha) - \{0\}} \frac{\theta}{\|\theta\|^2} H(d\theta) = \lim_{\varepsilon \downarrow 0} \lambda_\varepsilon \tag{2.17}$$

and write

$$\Lambda(H) = \{(\lambda, M) | (\lambda_0, M_0) \in \Delta_0\}, \tag{2.18}$$

where

$$\Delta_0 \equiv \bigcap_{\varepsilon \downarrow 0} \overline{\Delta}_\varepsilon. \tag{2.19}$$

Now define

$$\begin{aligned} \Xi = \{(\lambda, M, H, G, c) \in \mathbb{R}^p \times \mathscr{S}_p \times \mathscr{F}(\overline{\Theta}(\alpha) - \{0\}) \times \mathscr{L}f(\overline{\Theta}'(\alpha)) \\ \times \mathbb{R} | (\lambda, M) \in \Lambda(H)\} \end{aligned} \tag{2.20}$$

and for each $(\lambda, M, H, G, c) \in \Xi$, let

$$\begin{aligned} d(x) &\equiv d(x; \lambda, M, H, G, c) \\ &= \lambda^t l(x) + \frac{1}{2}\langle M_0, V(x)\rangle + \int_{\overline{\Theta}(\alpha) - \{0\}} \frac{R_\theta^{(1)}(x)}{\|\theta\|^2} H(d\theta) \\ &\quad + \int_{\overline{\Theta}'(\alpha)} R_\theta(x) G(d\theta) - c. \end{aligned} \tag{2.21}$$

Take Φ to be the class of tests ϕ such that for some $C \in \mathscr{C}$ and

$$(\lambda, M, H, G, c) \in \Xi - \{(0,0,0,0,0)\}, \tag{2.22}$$

$$|d(x)| < \infty \quad \text{for } x \in \operatorname{int} C \tag{2.23}$$

and

$$\phi(x) = \begin{cases} 1, & \text{if } x \notin C, \\ 1, & \text{if } d(x) > 0 \text{ and } x \in \operatorname{int} C, \\ 0, & \text{if } d(x) < 0 \text{ and } x \in \operatorname{int} C, \text{ a.e. } [\nu]. \end{cases} \tag{2.24}$$

Note that if (2.16) holds, then we can write

$$\begin{aligned} d(x) &= \lambda_0^t l(x) + \frac{1}{2}\langle M_0, V(x)\rangle \\ &\quad + \int_{\overline{\Theta}(\alpha) - \{0\}} \frac{R_\theta(x) - 1}{\|\theta\|^2} H(d\theta) + \int_{\overline{\Theta}'(\alpha)} R_\theta(x) G(d\theta) - c. \end{aligned} \tag{2.25}$$

It can be checked that the set Φ does not depend on the constant $\alpha > 0$. Thus if Θ is bounded, we can take α large enough that $\overline{\Theta}'(\alpha)$ is empty and G is unnecessary. If $\Theta - \{0\}$ is topologically separated from 0, then we can take α small enough that $\overline{\Theta}(\alpha) - \{0\} = \varnothing$, so that (λ, M, H) is unnecessary, and we can restrict c to be nonnegative. This latter situation is treated in Farrell (1968) and Ghia (1976). In Remark 2.5 we present an alternative characterization of Φ which, under conditions, does not need separate terms for $\Theta(\alpha)$ and $\Theta'(\alpha)$.

THEOREM 2.4. *Under Assumptions* 2.1–2.3, Φ *is an essentially complete class.*

PROOF. By Wald (1950), an essentially complete class consists of all weak* limits of Bayes tests, where each prior probability measure is concentrated on a finite number of points. Suppose $\{\pi_i\} \subseteq \mathscr{P}(\Theta)$ is a sequence of such priors, $\{\phi_i\}$ is a corresponding sequence of Bayes tests:

$$\phi_i(x) = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad \text{as} \int_{\Theta - \{0\}} f_\theta(x)\pi_i(d\theta) - \pi_i(\{0\})\begin{Bmatrix} > \\ < \end{Bmatrix}0 \tag{2.26}$$

and ϕ is a test which satisfies

$$\phi_i \to \phi \quad \text{in the weak* sense.} \tag{2.27}$$

We will show that $\phi \in \Phi$, which proves the theorem. Our main tool is to show that on a subsequence $\{i_0\}$,

$$\begin{aligned} &\phi_{i_0}(x) \to 1 \quad \text{if } x \in C^c \text{ or } x \in \operatorname{int} C \text{ and } d(x) > 0, \\ &\phi_{i_0}(x) \to 0 \quad \text{if } x \in \operatorname{int} C \text{ and } d(x) < 0, \end{aligned} \tag{2.28}$$

for some $C \in \mathscr{C}$ and d as in (2.21)–(2.23). Since $0 \le \phi(x) \le 1$ for all x, (2.28) implies that ϕ in (2.27) satisfies (2.24).

Now if $\pi_i(\{0\}) = 0$ infinitely often, it is easy to see that $\phi = 1$ a.e. Similarly, if $\pi_i(\{0\}) = 1$ infinitely often, then $\phi = 0$ a.e. In either case, $\phi \in \Phi$. From now on we assume that $0 < \pi_i(\{0\}) < 1$ for all i.

Let $(\bar{\pi}_i, \bar{G}_i, c_i) \in \mathscr{F}(\Theta(\alpha)) \times \mathscr{F}(\Theta'(\alpha)) \times \mathbb{R}$ be defined by

$$\bar{\pi}_i = a(\theta)^{-1}\pi_i I_{\Theta(\alpha)}, \qquad \bar{G}_i = a(\theta)^{-1}\pi_i I_{\Theta'(\alpha)}, \qquad c_i = \pi_i(\{0\}) - \bar{\pi}_i(\Theta(\alpha)).$$

Then we can rewrite the right-hand part of (2.26) as

$$\begin{aligned} &l^t(x)\int_{\Theta(\alpha)} \theta\bar{\pi}_i(d\theta) + \tfrac{1}{2}\left\langle \int_{\Theta(\alpha)} \theta\theta^t\bar{\pi}_i(d\theta), V(x)\right\rangle \\ &\quad + \int_{\Theta(\alpha)} R_\theta^{(2)}(x)\bar{\pi}_i(d\theta) + \int_{\Theta'(\alpha)} R_\theta(x)\bar{G}_i(d\theta) - c_i\begin{Bmatrix} > \\ < \end{Bmatrix}0. \end{aligned} \tag{2.29}$$

(See Assumption 2.2.) Now let

$$H_i = \begin{cases} \dfrac{\|\theta\|^2\bar{\pi}_i}{\displaystyle\int_{\Theta(\alpha)} \|\theta\|^2\bar{\pi}_i(d\theta)}, & \text{if } \bar{\pi}_i(\Theta(\alpha)) > 0, \\ \pi^*, & \text{if } \bar{\pi}_i(\Theta(\alpha)) = 0, \text{but } \Theta(\alpha) \ne \varnothing, \\ 0, & \text{if } \Theta(\alpha) = \varnothing, \end{cases} \tag{2.30}$$

where π^* is an arbitrarily chosen element of $\mathscr{P}(\Theta(\alpha))$. Thus

$$H_i = 0 \quad \text{if } \Theta(\alpha) = \varnothing, \qquad H_i \in \mathscr{P}(\Theta(\alpha)) \quad \text{if } \Theta(\alpha) \neq \varnothing. \tag{2.31}$$

We can now rewrite (2.29) as

$$\begin{aligned} &\alpha_i \left[l^t(x) \int_{\Theta(\alpha)} \frac{\theta}{\|\theta\|^2} H_i(d\theta) + \frac{1}{2} \left\langle \int_{\Theta(\alpha)} \frac{\theta\theta^t}{\|\theta\|^2} H_i(d\theta), V(x) \right\rangle \right. \\ &\qquad\qquad \left. + \int_{\Theta(\alpha)} \frac{R_\theta^{(2)}(x)}{\|\theta\|^2} H_i(d\theta) \right] \\ &\quad + \int_{\Theta'(\alpha)} R_\theta(x) G_i(d\theta) - \beta_i \left\{ \begin{matrix} > \\ < \end{matrix} \right\} 0, \end{aligned} \tag{2.32}$$

where

$$G_i \in \mathscr{F}(\Theta'(\alpha)) \quad \text{and} \quad (\alpha_i, \beta_i) \in \Gamma \equiv \{(r, s) | r \geq 0, \ r + |s| = 1\}. \tag{2.33}$$

There are two cases to consider, depending on the behavior of

$$\lambda_i(\alpha) \equiv \int_{\Theta(\alpha)} \frac{\theta}{\|\theta\|^2} H_i(d\theta). \tag{2.34}$$

CASE 1. $\limsup_{i \to \infty} \|\lambda_i(\alpha)\| = \infty$. Since $\|\lambda_i(\alpha) - \lambda_i(\varepsilon)\| \leq \varepsilon^{-1}$, $\limsup_{i \to \infty} \|\lambda_i(\varepsilon)\| = \infty$ for each $\varepsilon \in (0, \alpha)$. Thus we can find a subsequence of $\{i\}$, which we relabel $\{i\}$, such that

$$\left\| \lambda_i\left(\frac{1}{i}\right) \right\| \geq i \quad \text{and} \quad \frac{\left\| \lambda_i(\alpha) - \lambda_i\left(\frac{1}{i}\right) \right\|}{\left\| \lambda_i\left(\frac{1}{i}\right) \right\|} \leq \frac{1}{i} \tag{2.35}$$

for each i. Dividing both sides of (2.32) by $\alpha_i \|\lambda_i(1/i)\| + |\beta_i|$ yields

$$\gamma_i \left[l^t(x) \lambda_i\left(\frac{1}{i}\right) \Big/ \left\| \lambda_i\left(\frac{1}{i}\right) \right\| + Q \right] + \int_{\Theta'(\alpha)} R_\theta(x) G_i'(d\theta) - \delta_i \left\{ \begin{matrix} > \\ < \end{matrix} \right\} 0, \tag{2.36}$$

where

$$G_i' \in \mathscr{F}(\Theta'(\alpha)) \quad \text{and} \quad (\gamma_i, \delta_i) \in \Gamma \tag{2.37}$$

and

$$\begin{aligned} Q = \left\| \lambda_i\left(\frac{1}{i}\right) \right\|^{-1} &\left[l^t(x) \left(\lambda_i(\alpha) - \lambda_i\left(\frac{1}{i}\right) \right) \right. \\ &+ \frac{1}{2} \left\langle \int_{\Theta(\alpha)} \frac{\theta\theta^t}{\|\theta\|^2} H_i(d\theta), V(x) \right\rangle \\ &\left. + \int_{\Theta(\alpha)} \frac{R_\theta^{(2)}(x)}{\|\theta\|^2} H_i(d\theta) \right]. \end{aligned} \tag{2.38}$$

Now $Q = o(1)$ as $i \to \infty$ by (2.35), (2.31) and the fact that the two integrands are bounded [see (2.7) for the second integrand].

Now apply the Asymptotic Assumption 2.3 to $\{G_i'\}$, which yields a subsequence $\{i'\}$ for which (2.8), (2.10) and (2.11) are satisfied. By (2.9), we have $G \in \mathscr{L}f(\overline{\Theta}'(\alpha))$. We can find a further subsequence $\{i''\} \subseteq \{i'\}$, $\lambda^* \in \mathbb{R}^p$, with $\|\lambda^*\| = 1$ and $(\gamma, \delta) \in \Gamma$ such that

$$\frac{\lambda_{i''}(1/i'')}{\|\lambda_{i''}(1/i'')\|} \to \lambda^* \quad \text{and} \quad (\gamma_{i''}, \delta_{i''}) \to (\gamma, \delta) \tag{2.39}$$

(since Γ is compact). On this subsequence, we have by (2.10) and (2.39) that if $x \in \operatorname{int} C$, the limit of the left-hand side of (2.36) is

$$\gamma\lambda^{*t}l(x) + \int_{\Theta'(\alpha)} R_\theta(x)G(d\theta) - \delta, \tag{2.40}$$

bounded. If $x \notin C$, then (2.11) shows the limit to be ∞. Thus (2.11), (2.36) and (2.40) show that (2.23) and (2.28) hold, where d is given in (2.21) with

$$(\lambda, M, H, G, c) = (\gamma\lambda^*, 0, 0, G, \delta).$$

Since $\|\lambda^*\| = 1$ and $\gamma + |\delta| = 1$, $(\lambda, c) \neq (0,0)$, hence (2.22) holds.

To complete this part of the proof, we need to show that

$$(\lambda, 0) \in \Lambda(0). \tag{2.41}$$

Since (2.16) holds, by (2.18) it is enough to show that for each $\varepsilon > 0$,

$$(\lambda, 0) \in \overline{\Delta}_\varepsilon. \tag{2.42}$$

Define $H_k^* \in \mathscr{F}(\Theta(1/k))$ by

$$H_k^* = \left\|\lambda_k\left(\frac{1}{k}\right)\right\| H_k I_{\Theta(1/k)}$$

and let

$$(\lambda_k, M_k) = \int_{\Theta(1/k)} \frac{(\theta, \theta\theta^t)}{\|\theta\|^2} H_k^*(d\theta).$$

By (2.12), $(\lambda_k, M_k) \in \Delta_\varepsilon$ for $k > \varepsilon^{-1}$. By (2.39), $\lambda_{i''} \to \lambda^*$. Also, $\operatorname{tr} M_k = \|\lambda_k(1/k)\|^{-1} H_k(\Theta(1/k))$, hence by (2.31) and (2.35), $\operatorname{tr} M_{i''} \to 0$. Thus $(\lambda^*, 0) \in \overline{\Delta}_\varepsilon$, hence (2.42) holds since $\overline{\Delta}_\varepsilon$ is a cone.

CASE 2. $\limsup_{i\to\infty}\|\lambda_i(\alpha)\| < \infty$. Apply the Asymptotic Assumption 2.3 to $\{G_i\}$ in (2.32). Let

$$M_i(\varepsilon) = \int_{\Theta(\varepsilon)} \frac{\theta\theta^t}{\|\theta\|^2} H_i(d\theta) \tag{2.43}$$

and recall (2.34). If $\Theta(\alpha) = \varnothing$, then by (2.31) the bracketed term in (2.32) is 0, and we can take $\beta_i = 1$ since in (2.29), $c_i = \pi_i(\{0\}) > 0$. Thus (2.10) and (2.11) prove via (2.32) that along the sequence $\{i'\}$, (2.28) holds with $(\lambda, M, H, G, c) = (0, 0, 0, G, 1)$ in (2.21) and (2.22) and (2.23) hold.

If $\Theta(\alpha) \neq \varnothing$, (2.31) shows that we can take $\{H_i\}$ to be a subsequence in $\mathscr{P}(\overline{\Theta}(\overline{\alpha}))$ and that $\operatorname{tr} M_i(\alpha) = H_i(\Theta(\alpha)) = 1$. Also, $\limsup_{i\to\infty}\|\lambda_i(\alpha)\| < \infty$ and $(\alpha_i, \beta_i) \in \Gamma$. Thus there exist a subsequence $\{i''\} \subseteq \{i'\}$, $H^* \in \mathscr{P}(\overline{\Theta}(\alpha))$, $\lambda^* \in \mathbb{R}^p$, $M^* \in \mathscr{S}_p$ and $(\alpha^*, \beta^*) \in \Gamma$ such that

$$\begin{aligned} H_{i''} &\to H^* \quad \text{weakly}, \\ (\lambda_{i''}(\alpha), M_{i''}(\alpha)) &\to (\lambda^*, M^*), \\ (\alpha_{i''}, \beta_{i''}) &\to (\alpha^*, \beta^*). \end{aligned} \tag{2.44}$$

Let

$$H_0^* = H^* I_{\overline{\Theta}(\alpha) - \{0\}} \in \mathscr{F}(\overline{\Theta}(\alpha) - \{0\}). \tag{2.45}$$

Now for each $\varepsilon > 0$ with $H^*(\|\theta\| = \varepsilon) = 0$,

$$(\lambda_{i''}(\varepsilon), M_{i''}(\varepsilon)) \to (\lambda_\varepsilon^*(H_0^*), M_\varepsilon^*(H_0^*)) \tag{2.46}$$

from (2.13), since the integrands $\theta/\|\theta\|^2$ and $\theta\theta^t/\|\theta\|^2$ are bounded and continuous over $\overline{\Theta}(\varepsilon, \alpha)$. Thus by (2.12), the limit in (2.46) is in $\overline{\Delta}_\varepsilon$, hence $(\lambda^*, M^*) \in \Lambda(H_0^*)$ by (2.14).

Assumptions 2.1 and 2.2 show that $R_\theta^{(2)}(x)/\|\theta\|^2$ is bounded and continuous in $\overline{\Theta}(\alpha)$ and equals 0 when $\theta = 0$. Thus

$$\int_{\Theta(\alpha)} \frac{R_\theta^{(2)}(x)}{\|\theta\|^2} H_{i''}(d\theta) \to \int_{\overline{\Theta}(\alpha) - \{0\}} \frac{R_\theta^{(2)}(x)}{\|\theta\|^2} H_0^*(d\theta), \quad \text{bounded}. \tag{2.47}$$

Hence (2.5), (2.10), (2.44) and (2.47) show that the left-hand side of (2.32) approaches, along $\{i''\}$,

$$\begin{aligned} &\alpha^*\left[l^t(x)\lambda^* + \frac{1}{2}\langle M_0^*, V(x)\rangle + \int_{\overline{\Theta}(\alpha)-\{0\}} \frac{R_\theta^{(1)}(x)}{\|\theta\|^2} H_0^*(d\theta)\right] \\ &\quad + \int_{\overline{\Theta}'(\alpha)} R_\theta(x) G(d\theta) - \beta^*, \quad \text{bounded, if } x \in \operatorname{int} C \end{aligned} \tag{2.48}$$

and (2.11) shows that it approaches $+\infty$ if $x \in C^c$. Thus (2.28) holds with

$$(\lambda, M, H, G, c) = (\alpha^*\lambda^*, \alpha^* M^*, \alpha^* H_0^*, G, \beta^*)$$

in (2.21). Since $(\lambda^*, M^*) \in \Lambda(H^*)$, $\alpha^*(\lambda^*, M^*) \in \Lambda(\alpha^* H^*)$. Also, since $H_i \in \mathscr{P}(\overline{\Theta}(\alpha))$, $\operatorname{tr} M_i(\alpha) = 1$ by (2.43), hence

$$\operatorname{tr} M_0^* + H_0^*(\overline{\Theta}(\alpha) - \{0\}) = 1.$$

Thus $(M_0^*, H_0^*) \neq (0, 0)$, and since $\alpha^* + |\beta^*| = 1$, $(M_0, H, c) \neq (0, 0, 0)$, proving (2.22). This ends the proof of the theorem. □

The next result shows that the set Ξ in (2.20) is, in some sense, minimal. It is also useful when proving admissibility as in Section 3.

LEMMA 2.5. *Given $(\lambda, M, H, G, c) \in \Xi$, there exists a sequence $\{J_i\} \subseteq \mathscr{F}(\Theta)$ such that for each i,*

$$\int_{\Theta - \{0\}} f_\theta(x) J_i(d\theta) - J_i(\{0\})$$

$$(2.49) \qquad = l^t(x) \int_{\Theta(\alpha)} \frac{\theta}{\|\theta\|^2} H_i(d\theta) + \frac{1}{2} \left\langle \int_{\Theta(\alpha)} \frac{\theta\theta^t}{\|\theta\|^2} H_i(d\theta), V(x) \right\rangle$$

$$+ \int_{\Theta(\alpha)} \frac{R_\theta^{(2)}(x)}{\|\theta\|^2} H_i(d\theta) + \int_{\Theta'(\alpha)} R_\theta(x) G_i(d\theta) - c \equiv d_i(x),$$

where $H_i \in \mathscr{F}(\Theta(\alpha))$, $G_i \in \mathscr{F}(\Theta'(\alpha))$,

$$(2.50) \qquad \int_{\Theta(\alpha)} g(\theta) H_i(d\theta) \to \int_{\overline{\Theta}(\alpha) - \{0\}} g(\theta) H(d\theta)$$

for any continuous bounded function g with $g(0) = 0$,

$$(2.51) \qquad \int_{\Theta(\alpha)} \frac{(\theta, \theta\theta^t)}{\|\theta\|^2} H_i(d\theta) \to (\lambda, M)$$

and

$$(2.52) \qquad G_i \to G \quad \textit{vaguely}.$$

Assumptions 2.1 and 2.2 are enough to guarantee that for any x,

$$\int_{\Theta(\alpha)} \frac{R_\theta^{(2)}(x)}{\|\theta\|^2} H_i(d\theta) \to \int_{\bar{\theta}(\alpha) - \{0\}} \frac{R_\theta^{(2)}(x)}{\|\theta\|^2} H(d\theta).$$

Thus if we have that

$$\int_{\Theta'(\alpha)} R_\theta(x) G_i(d\theta) \to \int_{\overline{\Theta}'(\alpha)} R_\theta(x) G(d\theta),$$

then $d_i(x)$ of (2.49) approaches $d(x)$ of (2.21).

PROOF OF LEMMA 2.5. If $\Theta - \{0\}$ is topologically separated from 0, then we can choose α so that $\Theta(\alpha) = \varnothing$ and $c \geq 0$ as mentioned above Theorem 2.4. Since $G \in \mathscr{L}f(\overline{\Theta}'(\alpha))$ there exists a sequence $\{G_i\} \subseteq \mathscr{F}(\Theta'(\alpha))$ such that (2.52) holds, and the lemma follows by taking

$$J_i = a(\theta)^{-1} G_i + c\delta_0,$$

where δ_γ is the measure placing point mass 1 at γ.

Below we assume $\Theta - \{0\}$ is not topologically separated from 0. We can still take $\{G_i\}$ as above. Also, since $H \in \mathscr{F}(\overline{\Theta}(\alpha) - \{0\})$, there exists $\{H_i'\} \subseteq \mathscr{F}(\Theta(\alpha))$ such that

$$(2.53) \qquad H_i' \to H \quad \text{weakly}.$$

Since $(\lambda, M) \in \Lambda(H)$, (2.12) and (2.14) guarantee that for each $\varepsilon \in (0, \alpha]$, there exists a sequence $\{H_{i\varepsilon}\} \subseteq \mathscr{F}(\Theta(\varepsilon))$ such that $\int_{\Theta(\varepsilon)}(1/\|\theta\|^2)M_{i\varepsilon}(d\theta) < \infty$ for each i and

$$(2.54) \qquad \int_{\Theta(\varepsilon)} \frac{(\theta, \theta\theta^t)}{\|\theta\|^2} H_{i\varepsilon}(d\theta) \to (\lambda_\varepsilon, M_\varepsilon).$$

We can now choose a sequence $\{\varepsilon_i\} \subseteq (0, \alpha]$, $\varepsilon_i \downarrow 0$, such that (2.51) holds for a subsequence of $\{H_i^*\}$, where

$$(2.55) \qquad H_i^* = H_{i\varepsilon_i} + H_i' I_{\Theta(\varepsilon_i, \alpha)}.$$

We show that (2.50) holds, too.

Take g as in (2.50), and $\delta > 0$ with $H(\|\theta\| = \delta) = 0$. Then

$$\limsup_{i\to\infty} \left| \int_{\Theta(\alpha)} g(\theta) H_i^*(d\theta) - \int_{\overline{\Theta}(\alpha) - \{0\}} g(\theta) H(d\theta) \right|$$

$$(2.56) \qquad \le \limsup_{i\to\infty} \left| \int_{\Theta(\alpha)} g(\theta) H_i^*(d\theta) - \int_{\overline{\Theta}(\delta, \alpha)} g(\theta) H(d\theta) \right| + \int_{\overline{\Theta}(\delta)} |g(\theta)| H(d\theta)$$

$$\le (\operatorname{tr} M) \sup_{\theta \in \Theta(\delta)} |g(\theta)| + \int_{\overline{\Theta}(\delta)} |g(\theta)| H(d\theta),$$

since (2.51) implies that $\lim_{i\to\infty} H_i^*(\Theta(\alpha)) = \operatorname{tr} M$ and (2.53) and (2.55) imply that

$$\lim_{i\to\infty} \left| \int_{\Theta(\delta, \alpha)} g(\theta) H_i^*(d\theta) - \int_{\overline{\Theta}(\delta, \alpha)} g(\theta) H(d\theta) \right| = 0.$$

Since δ can be chosen arbitrarily close to 0, the limit superior in (2.56) equals 0, proving (2.50).

Now since $0 \in \bar{\theta}$, there exists a sequence $\{\theta_i\} \subseteq \Theta(\alpha)$ such that $\|\theta_i\| \to 0$. Let

$$(2.57) \qquad H_i = H_i^* + \|\theta_i\|^2 (-c)^+ \delta_{\|\theta_i\|},$$

where $z^+ \equiv \max(z, 0)$. It is clear that (2.50) and (2.51) also hold for $\{H_i\}$. Define

$$(2.58) \qquad c_i = c + \int_{\Theta(\alpha)} \frac{1}{\|\theta\|^2} H_i(d\theta).$$

By (2.57), $c_i \ge 0$. Thus the lemma follows by setting

$$(2.59) \qquad J_i = a(\theta)^{-1} \left[\|\theta\|^2 H_i + G_i \right] + c_i \delta_0. \qquad \square$$

We conclude this section with some remarks.

REMARK 2.6. Suppose $a^*(\theta)$ is another function which satisfies Assumptions 2.1 and 2.2 and let $R_\theta^*(x)$, etc., be the corresponding functions from (2.1) and

(2.3)–(2.5). Then we can rewrite d in (2.21) as

$$d(x) = \lambda^{*t}l(x) + \langle M_0^*, V^*(x)\rangle + \int_{\Theta(\alpha)-\{0\}} \frac{R_\theta^{*(1)}(x)}{\|\theta\|^2} H^*(d\theta) + \int_{\overline{\Theta}'(\alpha)} R_\theta^*(x) G^*(d\theta) - c^*, \tag{2.60}$$

where

$$H^* = [a(\theta)/a^*(\theta)]H, \qquad G^* = [a(\theta)/a^*(\theta)]G, \qquad (\lambda^*, M^*) \in \Lambda(H^*)$$

and c^* is given below.

To prove (2.60), we first note that the final integrals in (2.60) and (2.21) are equal. In Lemma 2.5 we present a sequence $\{H_i\} \in \mathscr{L}(\Theta(\alpha))$ such that

$$\lim_{i\to\infty} \int_{\Theta(\alpha)} \frac{R_\theta(x) - 1}{\|\theta\|^2} H_i(d\theta) = \lambda^t l(x) + \frac{1}{2}\langle M_0, V(x)\rangle + \int_{\overline{\Theta}(\alpha)-\{0\}} \frac{R_\theta^{(1)}(x)}{\|\theta\|^2} H(d\theta). \tag{2.61}$$

If we let $H_i^* = [a(\theta)/a^*(\theta)]H_i$, then

$$\int_{\Theta(\alpha)} \frac{R_\theta(x) - 1}{\|\theta\|^2} H_i(d\theta) = \int_{\Theta(\alpha)} \frac{R_\theta^*(x) - 1}{\|\theta\|^2} H_i^*(d\theta) + b_i^*, \tag{2.62}$$

where

$$b_i^* = \int_{\Theta(\alpha)} \frac{a(\theta)/a^*(\theta) - 1 - \theta^t l_0}{\|\theta\|^2} H_i(d\theta) + l_0^t \int_{\Theta(\alpha)} \frac{\theta}{\|\theta\|^2} H_i(d\theta), \tag{2.63}$$

where

$$l_0 = \left\{ \frac{\partial}{\partial\theta_i} \frac{a(\theta)}{a^*(\theta)} \right\}_{i=1}^p \Bigg|_{\theta=0}.$$

Assumption 2.2 holds for R_θ and R_θ^*, hence for $R_\theta/R_\theta^* \equiv a(\theta)/a^*(\theta)$. Thus the limit of the first term on the right-hand side of (2.62) is the sum of the first three terms on the right-hand side of (2.60) as in (2.61). Also, the limit of b_i^* in (2.63) is finite. Thus (2.60) holds with $c^* = c - \lim_{i\to\infty} b_i^*$. Now take ϕ in (2.24) and suppose int C is nonempty. With $x_0 \in$ int C and $a^*(\theta) = f_\theta^{-1}(x_0)$, we have

$$a(\theta)/a^*(\theta) = R_\theta(x_0),$$

hence by definition of G^*, and since (2.23) holds,

$$G^*(\overline{\Theta}'(\alpha)) < \infty, \tag{2.64}$$

which occurs, e.g., in the exponential family case of Section 5. Then we can combine the $\Theta(\alpha)$ and $\Theta'(\alpha)$ terms in (2.60) to obtain

$$d(x) = \lambda^{**t}l^*(x) + \langle M_0^*, V^*(x)\rangle + \int_{\overline{\Theta}-\{0\}} \frac{R_\theta^{*(1)}(x)}{\|\theta\|^2} H^{**}(d\theta) + (\text{const.}), \tag{2.65}$$

where $H^{**} = H^* + \|\theta\|^2 G^*$ and $\lambda^{**} = \lambda^* + \int_{\overline{\theta}'(\alpha)} \theta G^*(d\theta)$, so that $(\lambda^{**}, M^{**}) \in \Lambda(H^{**})$ defined with $\alpha = \infty$ and $M_0^{**} = M_0^*$.

REMARK 2.7. Suppose the testing problem (1.1) is invariant under a finite group K. It is easy to obtain Φ_K, an essentially complete class of K-invariant tests, from Theorem 2.4. Take

$$\overline{R}_\theta(x) = \frac{1}{\#K} \sum_{g\in K} R_\theta(gx). \tag{2.66}$$

Then $\overline{R}_\theta(x)$ satisfies Assumptions 2.1–2.3 if $R_\theta(x)$ does. The local terms (2.3) and (2.4) are

$$\bar{l}(x) = \frac{1}{\#K} \sum_{g\in K} l(gx) \quad \text{and} \quad \overline{V}(x) = \frac{1}{\#K} \sum_{g\in K} V(gx), \tag{2.67}$$

and the set $\overline{\mathscr{C}}$ consists of the invariant sets in $\mathscr{C}$. Then Φ_K is given as in (2.24) with the barred quantities, where H and G can be taken to be invariant measures.

REMARK 2.8. It may happen that $l(x) \equiv 0$. For an example see the problem of testing sphericity of a covariance matrix in Cohen and Marden (1988). In such cases, Theorem 2.4 is useless since by taking $(\lambda, M, H, G, c) = (\lambda, 0, 0, 0, 0)$ for any $\lambda \neq 0$, we see that any test ϕ is in Φ. However, a slight modification of the proof of the theorem shows that when $l(x) \equiv 0$, we can replace (2.22) with

$$(M, H, G, c) \in \Lambda_{20} \times \mathscr{F}(\overline{\Theta}(\alpha) - \{0\}) \times \mathscr{L}f(\overline{\Theta}'(\alpha)) \times R. \tag{2.68}$$

Here, Λ_{20} is the projection of Δ_0 in (2.19) on the M-space. We now show how this result can be proved.

Follow the proof of Theorem 2.4 until (2.33). At this point, the behavior of $\lambda_i(\alpha)$ in (2.34) is irrelevant since $l(x) \equiv 0$. Now skip to line (2.43). Proceed through to the end of the proof eliminating the "λ" terms, replacing $\overline{\Delta}_\varepsilon$ by $\overline{\Delta}_{2\varepsilon}$, where $\Delta_{2\varepsilon}$ is the projection of Δ_ε on the M-space and replacing $\Lambda(H_0^*)$ and $\Lambda(H^*)$ by Δ_{20}.

3. Admissibility and the complete class. In this section we present conditions on a test $\phi \in \Phi$ adequate to prove that

$$r_\theta(\phi') \leq r_\theta(\phi) \quad \forall\, \theta \Rightarrow \phi = \phi' \text{ a.e. for any test } \phi'. \tag{3.1}$$

Any such test ϕ is clearly admissible. In addition, if all tests in Φ satisfy (3.1), then Φ is a complete (in fact, the minimal complete) class: Take ϕ' admissible.

By Theorem 2.4, Φ is essentially complete, hence there exists $\phi \in \Phi$ such that $r_\theta(\phi) = r_\theta(\phi')\ \forall\ \theta$. But then (3.1) shows that $\phi = \phi'$ a.e., hence $\phi' \in \Phi$ since Φ is defined only up to null sets.

We present an additional assumption.

ASSUMPTION 3.1. The test $\phi \in \Phi$ with its attendant $C \in \mathscr{C}$ and $(\lambda, M, H, G, c) \in \Xi$ satisfies the following:

(i) The set C has the property that for any ψ, ψ',

$$\psi \in \Phi(C) \quad \text{and} \quad r_\theta(\psi') \le r_\theta(\psi) \quad \forall\ \theta \Rightarrow \psi' \in \Phi(C), \tag{3.2}$$

where $\Phi(C)$ consists of all tests ϕ' such that

$$\nu(\{x|\phi'(x) < 1\} \cap C^c) = 0. \tag{3.3}$$

(ii) There exists a sequence $\{J_i\} \subseteq \mathscr{F}(\Theta)$ such that

$$d_i(x) \equiv \int_{\Theta - \{0\}} R_\theta(x) J_i(d\theta) - J_i(\{0\}) \to d(x) \qquad \text{for } x \in \operatorname{int} C \tag{3.4}$$

and

$$\lim_{i \to \infty} \int_{\operatorname{int} C} (\phi(x) - \phi_i(x))\, d_i(x) \nu(dx) = 0, \tag{3.5}$$

where $\phi_i \in \Phi(C)$ is Bayes with respect to J_i among tests in $\Phi(C)$:

$$\phi_i(x) = \begin{cases} 1, & \text{if } x \notin C, \\ 1, & \text{if } d_i(x) > 0 \text{ and } x \in \operatorname{int} C, \\ 0, & \text{if } d_i(x) < 0 \text{ and } x \in \operatorname{int} C. \end{cases} \tag{3.6}$$

(iii) $\nu(\partial C) = 0$.
(iv) $\nu(\{x|d(x) = 0\}) = 0$.

The requirement (i) is a fairly strong one on C. It implies that if a test in $\Phi(C)$ is admissible among tests in $\Phi(C)$, then it is admissible among all tests. Thus, for example, the test $1 - I_C$ is admissible since any essentially different test in $\Phi(C)$ will have strictly greater risk at $\theta = 0$.

The next result shows that (ii)–(iv) are sufficient to apply Blyth's (1951) method to show that ϕ is an admissible limit of Bayes tests in $\Phi(C)$.

LEMMA 3.2. *Suppose Assumptions* 2.1–2.3 *hold, and* $\phi \in \Phi$ *satisfies Assumption* 3.1. *Then* ϕ *satisfies* (3.1).

PROOF. Suppose that for the test ϕ',

$$r_\theta(\phi') \le r_\theta(\phi) \quad \forall\ \theta. \tag{3.7}$$

Thus by Assumption 3.1(i),

$$\mu(\{x|\phi'(x) \ne \phi(x)\} \cap C^c) = 0. \tag{3.8}$$

Using the sequence $\{J_i\}$ in (ii), we have

$$
\begin{aligned}
0 &\geq \liminf_{i\to\infty} \int_\Theta (r_\theta(\phi') - r_\theta(\phi)) J_i(d\theta) \quad \text{[by (3.7)]} \\
&= \liminf_{i\to\infty} \int_{\text{int}\, C} (\phi(x) - \phi'(x)) d_i(x) \nu(dx) \quad \text{[by (3.8) and (iii)]} \\
&\geq \liminf_{i\to\infty} \int_{\text{int}\, C} (\phi(x) - \phi_i(x))\, d_i(x)\nu(dx) \\
&\quad + \liminf_{i\to\infty} \int_{\text{int}\, C} (\phi_i(x) - \phi'(x))\, d_i(x)\nu(dx) \\
&\geq \int_{\text{int}\, C} \liminf_{i\to\infty} (\phi_i(x) - \phi'(x))\, d_i(x)\nu(dx) \quad \text{[by (3.5) and Fatou's lemma]} \\
&= \int_{\text{int}\, C} (\phi(x) - \phi'(x))\, d(x)\nu(dx) \quad \text{[by (3.4) and (3.6)]}.
\end{aligned} \tag{3.9}
$$

The application of Fatou's Lemma is valid since by (3.6), the integrand is nonnegative a.e. on int C. By (2.24), the final integrand in (3.9) is nonnegative, hence for (3.9) to hold it must be that

$$\nu(\{x \mid \phi'(x) \neq \phi(x)\} \cap \{x \mid d(x) \neq 0\} \cap \text{int}\, C) = 0. \tag{3.10}$$

Thus (3.8), (3.10), (iii) and (iv) prove that $\phi = \phi'$ a.e. □

4. More on $\Lambda(H)$. Lemma 2.5 shows that $\Lambda(H)$ of (2.14) is the minimal subset of $\mathbb{R}^p \times \mathscr{S}_p$ which we need for Theorem 2.5. In this section we give some characterizations of the sets $\Lambda(H)$ and Δ_0 or at least sets which are only slightly larger.

In Section 2 we showed that (2.16) implies (2.18). If (2.16) fails, it may still be that on a sequence $\{\varepsilon_i\}$, $\varepsilon_i \downarrow 0$, for $(\lambda, M) \in \Lambda(H)$,

$$\lambda_{\varepsilon_i} \to \lambda_0 \tag{4.1}$$

for some $\lambda_0 \in \mathbb{R}^p$. For any such limit we must have that $(\lambda_0, M_0) \in \Delta_0$ as defined in (2.19). On the other hand, if

$$\|\lambda_{\varepsilon_i}\| \to \infty, \tag{4.2}$$

then we can look at the sequence $(\lambda_{\varepsilon_i}, M_{\varepsilon_i})/\|\lambda_{\varepsilon_i}\|$. Since $\operatorname{tr} M_{\varepsilon_i} \leq \operatorname{tr} M < \infty$ for all i, on any subsequence for which $\lambda_{\varepsilon_i}/\|\lambda_{\varepsilon_i}\|$ has a limit, say λ'_0, $(\lambda'_0, 0) \in \Delta_0$ since $(\lambda_{\varepsilon_i}, M_{\varepsilon_i})/\|\lambda_{\varepsilon_i}\| \in \Delta_{\varepsilon_i}$ for each i. Thus the set Δ_0 is central to the very local structure of $\Lambda(H)$.

For $\varepsilon \geq 0$, let $\Delta_{1\varepsilon} \subseteq \mathbb{R}^p$ and $\Delta_{2\varepsilon} \subseteq \mathscr{S}_p$ be the projections of Δ_ε on the λ- and M-components, respectively. Then

$$\Delta_\varepsilon \subseteq \Delta_{1\varepsilon} \times \Delta_{2\varepsilon}. \tag{4.3}$$

Note that $\Delta_{1\varepsilon} = K^*(\Theta(\varepsilon))$, where for a set $\Omega \subseteq \mathbb{R}^p$,

$$
\begin{aligned}
K(\Omega) &= \text{smallest cone containing } \Omega; \\
K^*(\Omega) &= \text{smallest convex cone containing } \Omega.
\end{aligned} \tag{4.4}
$$

Thus if $(\lambda, M) \in \Lambda(H)$, by (2.12), $\lambda_\varepsilon \in \overline{K}^*(\Theta(\varepsilon))$, hence

$$\Delta_{10} \subseteq \bigcap_{\varepsilon \downarrow 0} \overline{K}^*(\Theta(\varepsilon)) \equiv K_0^*. \tag{4.5}$$

Next, consider

$$\mathscr{M}(\Omega) \equiv \left\{ \int_\Omega \frac{\theta\theta^t}{\|\theta\|^2} H(d\theta) | H \in \mathscr{F}(\Omega - \{0\}) \right\}. \tag{4.6}$$

As above, we have that

$$\Delta_{20} \subseteq \bigcap_{\varepsilon \downarrow 0} \overline{\mathscr{M}}(\Theta(\varepsilon)). \tag{4.7}$$

Since the integral in (4.6) depends on θ only through $\theta/\|\theta\|$, we have that

$$\mathscr{M}(\Omega) = \mathscr{M}(K(\Omega)) = \mathscr{M}(K(\Omega) \cap \{\theta | \|\theta\| = 1\}). \tag{4.8}$$

Using the following lemma, we can prove that

$$\overline{\mathscr{M}}(K(\Omega)) = \mathscr{M}(\overline{K}(\Omega)). \tag{4.9}$$

LEMMA 4.1. *If K is a closed cone, then $\mathscr{M}(K)$ is closed.*

PROOF. By (4.8) we have that $\mathscr{M}(K) = \mathscr{M}(K \cap \{\theta | \|\theta\| = 1\})$ and, since K is closed, that $K \cap \{\theta | \|\theta\| = 1\}$ is compact. Thus from (4.6), $\mathscr{M}(K)$ is closed. □

An immediate corollary of this result is that if $K_1 \supseteq K_2 \supseteq \cdots$ is a sequence of closed cones, then $\bigcap \mathscr{M}(K_i) = \mathscr{M}(\bigcap K_i)$. Hence (4.7)–(4.9) show that

$$\Delta_{20} \subseteq \mathscr{M}(K_0), \tag{4.10}$$

where

$$K_0 \equiv \bigcap_{\varepsilon \downarrow 0} \overline{K}(\Theta(\varepsilon)). \tag{4.11}$$

Often, Δ_{20} can be given more precisely than in (4.10). For a cone K, let

$$K_2 = K \cap -K = \{\theta | \theta \in K \text{ and } -\theta \in K\}. \tag{4.12}$$

We will show that

$$\Delta_{20} \subseteq \mathscr{M}(K_{02}^*), \tag{4.13}$$

where $K_{02}^* = \bigcap_{\varepsilon \downarrow 0} K_2^*(\Theta(\varepsilon))$; cf. (4.5). Thus using (4.10), (4.13) and Lemma 4.1 show that $\Delta_{20} \subseteq \mathscr{M}(K_0 \cap K_{02}^*)$, hence by (4.3) and (4.5),

$$\Delta_0 \subseteq K_0^* \times \mathscr{M}(K_0 \cap K_{02}^*). \tag{4.14}$$

The inclusion in (4.14) can be strict. See Example 6.1. The verification of (4.13) will follow the next two lemmas.

LEMMA 4.2. *If K is a closed convex cone in $\mathbb{R}^p$, then there exists a $\gamma_0 \in \mathbb{R}^p$ such that*

$$K \subseteq \{\theta | \gamma_0^t \theta \le 0\} \quad \text{and} \quad K_2 = K \cap \{\theta | \gamma^t \theta_0 = 0\}, \tag{4.15}$$

and $\gamma_0 = 0$ if and only if $K = \mathbb{R}^p$.

PROOF. First, notice that K_2 is a hyperplane of dimension q, $0 \le q \le p$, which passes through 0. If $q = p$, then $K_2 = K = \mathbb{R}^p$, so that (4.15) holds with $\gamma_0 = 0$. If $q < p$, then we can rotate K so that

$$K = K_1 \times \mathbb{R}^q \quad \text{and} \quad K_2 = \{0\} \times \mathbb{R}^q, \tag{4.16}$$

where $K_1 \subseteq \mathbb{R}^{p-q}$ is a closed convex cone which is pointed, that is,

$$\theta_1 \in K_1 - \{0\} \Rightarrow -\theta_1 \notin K_1.$$

Thus there exists a vector $\gamma_{10} \in \mathbb{R}^q - \{0\}$ such that $\gamma_{10}^t \theta_1 < 0$ for all $\theta_1 \in K_1 - \{0\}$. (Let $-\gamma_{10}$ be any vector in the relative interior of K_1.) Now take $\gamma_0 = \binom{\gamma_{10}}{0}$ in (4.15). □

LEMMA 4.3. *If for some $\gamma \in \mathbb{R}^p$ and $\delta > 0$, $\Theta(\delta) \subseteq \{\theta | \gamma^t \theta \le 0\}$, then*

$$\Delta_{20} \subseteq \mathscr{M}\left(K_0 \cap \{\theta | \gamma^t \theta = 0\}\right). \tag{4.17}$$

PROOF. If $\gamma = 0$, (4.17) is (4.10). Otherwise, take $(\lambda, M) \in \Lambda(H)$ and $\alpha = \delta$. For $\varepsilon \in (0, \delta)$, by (2.13) and since $\gamma^t \theta \le 0$,

$$\gamma^t \lambda = \gamma^t \lambda_\varepsilon + \int_{\overline{\Theta}(\varepsilon, \delta)} \frac{\gamma^t \theta}{\|\theta\|^2} H(d\theta) \le \gamma^t \lambda_\varepsilon. \tag{4.18}$$

Since $(\lambda_\varepsilon, M_\varepsilon) \in \overline{\Delta}_\varepsilon$, we have $\{H'_{\varepsilon i}\} \subseteq \mathscr{F}(\Theta(\varepsilon))$ such that (2.16) holds. Take $\beta < 0$ and let

$$A_\beta = \left\{\theta | \gamma^t \frac{\theta}{\|\theta\|} < \beta\right\}. \tag{4.19}$$

Then by (4.18),

$$\gamma^t \lambda \le \gamma^t \lambda_\varepsilon \le \liminf_{i \to \infty} \int_{A_\beta \cap \Theta(\varepsilon)} \frac{\gamma^t \theta}{\|\theta\|^2} H'_{\varepsilon i}(d\theta) \le \frac{\beta}{\varepsilon} \limsup_{i \to \infty} H'_{\varepsilon i}\left(\Theta(\varepsilon) \cap A_\beta\right) \tag{4.20}$$

and

$$\begin{aligned} M_\varepsilon &= \lim_{i \to \infty} \int_{\Theta(\varepsilon) \cap A_\beta} \frac{\theta \theta^t}{\|\theta\|^2} H'_{\varepsilon i}(d\theta) + \lim_{i \to \infty} \int_{\Theta(\varepsilon) \cap A_\beta^c} \frac{\theta \theta^t}{\|\theta\|^2} H'_{\varepsilon i}(d\theta) \\ &\equiv M_{\varepsilon 1} + M_{\varepsilon 2}, \end{aligned} \tag{4.21}$$

where we may need to take a subsequence on the right-hand side of (4.21). Now from (4.20) and (4.21),

$$\operatorname{tr} M_{\varepsilon 1} = \lim_{i \to \infty} H_{\varepsilon i}\big(\Theta(\varepsilon) \cap A_\beta\big) \le \frac{\varepsilon}{\beta} \gamma^t \lambda \tag{4.22}$$

(recall $\alpha > 0$ and $\gamma^t\lambda \le 0$). Thus $\operatorname{tr} M_{\varepsilon 1} \to 0$ as $\varepsilon \to 0$, so that

$$M_0 = \lim_{\varepsilon \downarrow 0} M_\varepsilon = \lim_{\varepsilon \downarrow 0} M_{\varepsilon 2}. \tag{4.23}$$

But $M_{\varepsilon 2} \in \mathscr{M}(\overline{K}(\theta(\varepsilon)) \cap A_\beta^c)$ since A_β^c is a closed cone, hence by Lemma 4.1 and (4.11),

$$M_0 \in \mathscr{M}\big(K_0 \cap A_\beta^c\big). \tag{4.24}$$

Finally, take the intersection over $\beta > 0$. Since $\bigcap_{\beta < 0} A_\beta = \{\theta | \gamma^t\theta \ge 0\}$ and $K_0 \subseteq \overline{K}(\theta(\delta)) \subseteq \{\theta | \gamma^t \theta \le 0\}$, we obtain

$$M_0 \in \mathscr{M}\big(K_0 \cap \{\theta | \gamma^t\theta = 0\}\big),$$

proving the lemma. □

Turn to (4.13). If $K_0^* = \mathbb{R}^p$, then $K_{02}^* = \mathbb{R}^p$ and (4.13) is trivial. Suppose $K_0^* \ne \mathbb{R}^p$. Take $\gamma_0 \ne 0$ from Lemma 4.2 which corresponds to K_0^*. For $\delta > 0$ small enough we must have that

$$\overline{K}^*(\Theta(\delta)) \subseteq \{\theta | \gamma_0^t \theta \le 0\}. \tag{4.25}$$

Apply Lemma 4.3 to show that (4.17) holds with $\gamma = \gamma_0$. But by (4.15), $K_{02}^* = K_0^* \cap \{\theta | \gamma_0^t\theta = 0\}$, hence (4.13) follows from (4.17).

We end this section with some examples.

EXAMPLE 4.4 (Θ is locally $\mathbb{R}^p$). Suppose that for some $\delta > 0$, $\Theta(\delta) \cup \{0\}$ is an open neighborhood of 0 in $\mathbb{R}^p$. Then for any H, $\Lambda(H)$ is unrestricted in the sense that

$$\Lambda(H) = \{(\lambda, M) | \lambda \in \mathbb{R}^p \text{ and } M_0 \in \mathscr{S}_p\}. \tag{4.26}$$

The inclusion " $\subseteq$ " in (4.26) holds always. The inclusion " $\supseteq$ " will hold by (2.14) if we show that for any $\varepsilon > 0$,

$$\overline{\Delta}_\varepsilon = \mathbb{R}^p \times \mathscr{S}_p. \tag{4.27}$$

Take $(\lambda^*, M^*) \in \mathbb{R}^p \times \mathscr{S}_p$ and let $H, G \in \mathscr{F}(\Theta(\delta))$ satisfy

$$\begin{aligned} &\int_{\Theta(\delta)} \theta H(d\theta) = \lambda^*, \qquad \int_{\Theta(\delta)} \theta\theta^t H(d\theta) = K^*, \\ &\int_{\Theta(\delta)} \theta G(d\theta) = 0, \qquad \int_{\Theta(\delta)} \theta\theta^t G(d\theta) = M^*, \end{aligned} \tag{4.28}$$

where K^* is arbitrary. Define

$$H_i(d\theta) = \|\theta\|^2 i H(i d\theta) + \|\theta\|^2 i^2 G(i d\theta) \in \mathscr{F}(\Theta(\delta / i)),$$

so that

$$(4.29)\qquad \int_{\Theta(\delta/i)} \frac{(\theta, \theta\theta^t)}{\|\theta\|^2} H_i(d\theta) = (\lambda^*, M^* + (1/i)K^*) \in \Delta_\varepsilon.$$

Hence $(\lambda^*, M^*) \in \Delta_\varepsilon$, proving (4.27).

EXAMPLE 4.5 (Θ is locally pointed). Assume that for some $\delta > 0$ $K^*(\Theta(\delta))$ is pointed, so that for some $\beta < 0$ and $\gamma_0 \neq 0$,

$$(4.30)\qquad \frac{\gamma_0^t\theta}{\|\theta\|} < \beta \quad \text{for } \theta \in \Theta(\delta).$$

Then $K_{02} = \{0\}$ so that $\Delta_{20} = \{0\}$ by (4.14). Also, from (4.18), for $(\lambda, M) \in \Lambda(H)$,

$$(4.31)\qquad \gamma_0^t\lambda \le \int_{\overline{\Theta}(\varepsilon, \alpha)} \frac{\gamma_0^t\theta}{\|\theta\|^2} H(d\theta) \le \beta \int_{\overline{\Theta}(\varepsilon, \alpha)} \frac{1}{\|\theta\|} H(d\theta).$$

Since $\gamma_0^t\lambda \le 0$ and $\beta < 0$, (4.31) shows that (2.16) holds. Hence $\Delta_{10} = K_0^*$ in (4.5), and we have equality in (4.14), so that $\Delta_0 = K_0^* \times \{0\}$ and by (2.18), $\Lambda(H) = \{(\lambda, M) | \lambda \in K_0^*, M_0 = 0\}$. We can now write d as in (2.25) without the $\langle M_0, V(x)\rangle$ term, which is the characterization given in Marden (1982).

EXAMPLE 4.6 (Θ is locally one-sided). Suppose that for some $\delta > 0$,

$$K_0^*(\Theta(\delta)) = K_1 \times \mathbb{R}^q$$

as in (4.16) with $0 < q < p$. Examples 4.4 and 4.5 treat $q = p$ and $q = 0$, respectively. As in Example 4.5, if $(\lambda, M) \in \Lambda(H)$, then

$$M_{0ii} = 0 \quad \text{and} \quad \int_{\overline{\Theta}(\alpha) - \{0\}} \frac{\theta_i}{\|\theta\|^2} H(d\theta) < \infty \text{ for } i = 1, \dots, p - q,$$

while as in Example 4.4, $(\lambda^{(2)}, M_0^{(2)})$ ranges freely over $\mathbb{R}^q \times \mathscr{S}_q$, where $\lambda^{(2)}$ is the lower $q \times 1$ subvector of λ and $M_0^{(2)}$ is the lower right $q \times q$ submatrix of M_0.

EXAMPLE 4.7 (Θ is locally the axes). Suppose that for some $\delta > 0$, $\Theta(\delta) = \{\theta | \|\theta\| \le \delta$ and exactly one component of θ is nonzero$\}$. Then K_0 in (4.11) is the set of axes in $\mathbb{R}^p$, while $K_0^* = \mathbb{R}^p$. Then $\Lambda(H) = \{(\lambda, M) | \lambda \in \mathbb{R}^p$ and $M_0 \in \mathscr{S}_p^D\}$, where $\mathscr{S}_p^D$ is the set of diagonal matrices in $\mathscr{S}_p$.

5. Exponential families. Assume we have an exponential family as in (1.3) and take $a(\theta) = \exp\{\psi(\theta)\}$ in (2.1) so that

$$(5.1)\qquad R_\theta(x) = \exp\{\theta^t x\}.$$

It is easy to show that Assumptions 2.1 and 2.2 hold with $l(x) = x$ and $V(x) = xx^t$. A result in Brown [(1986), Lemma 7.17] shows that the Asymptotic

Assumption 2.3 holds where $\mathscr{C}$ consists of all sets $C \subseteq \mathscr{X}$ which satisfy

$$C = \bigcap_{\gamma \in K_1 \cap \{\theta | \|\theta\| = 1\}} \{x | \gamma^t x \le \sup\{\gamma^t y | y \in C\}\}, \tag{5.2}$$

where

$$K_1 \equiv \bigcap_{N > 0} \bar{K}(\Theta'(N)) \tag{5.3}$$

[see (4.4)].

Note that every set in $\mathscr{C}$ is closed and convex. If Θ is a cone, then $C \in \mathscr{C}$ if and only if it also satisfies the monotonicity condition

$$x \in C \quad \text{and} \quad y \in \mathscr{X} \text{ with } \theta^t y \le \theta^t x \ \forall\, \theta \in \Theta \Rightarrow y \in C. \tag{5.4}$$

See Birnbaum (1955) and Eaton (1970). If $\Theta = \mathbb{R}^p$, the condition (5.4) is vacuous, hence $\mathscr{C}$ consists of all closed convex sets.

Turn to Remark 2.6. Suppose for test (2.24), int C is nonempty and take $x_0 \in \operatorname{int} C$. Then with $a^*(\theta) = \exp\{-\theta^t x_0\}$, we have that (2.53) and (2.54) hold. Hence we can write d in (2.25) as

$$\begin{aligned} d(x) &= \lambda^{**t}(x - x_0) + \frac{1}{2}\left\langle M_0^*, (x - x_0)(x - x_0)^t \right\rangle \\ &\quad + \int_{\bar{\Theta} - \{0\}} \frac{\exp\{\theta^t(x - x_0)\} - 1 - \theta^t(x - x_0)}{\|\theta\|^2} H^{**}(d\theta) - c^*. \end{aligned} \tag{5.5}$$

Finally, let $\mathscr{C}^*$ be the class of sets $C \in \mathscr{C}$ which satisfy (3.2). Define $\mathscr{W}$ to be the subset of $\{\theta \in \mathbb{R}^p | \|\theta\| = 1\} \times [-\infty, \infty]$ such that for each $(\lambda, c) \in \mathscr{W}$, there exists a sequence $\{\theta^{(i)}\} \subseteq \Theta$ and $\tau \ge 0$ such that

$$\begin{aligned} &\text{(i)} && \|\theta^{(i)}\| \to \infty \quad \text{as } i \to \infty, \\ (5.6)\ &\text{(ii)} && \|\theta^{(i)} - \|\theta^{(i)}\|\lambda\| \le \tau \quad \text{for all } i, \\ &\text{(iii)} && \int_{D(\lambda, c)} \exp\{\tau\|x\|\}\nu(dx) < \infty, \end{aligned}$$

where

$$D(\lambda, c) = \{x | \lambda^t x \le c\}. \tag{5.7}$$

Thus (5.6)(i) and (ii) imply that

$$\frac{\theta^{(i)}}{\|\theta^{(i)}\|} \to \lambda. \tag{5.8}$$

We remark that if $(\lambda, c) \in \mathscr{W}$, then $(\lambda, c') \in \mathscr{W}$ for $c < c'$; if the sequence $\{\theta^{(i)} - \|\theta^{(i)}\|\lambda\}$ is contained in a compact subset of the natural parameter space (1.4), $(\lambda, c) \in \mathscr{W}$ for all c and if Θ is a closed cone, $\mathscr{W} = \Theta \times [-\infty, \infty]$.

Define $\mathscr{C}^{**}$ to be the class of all sets of the form

$$C = \bigcap_{(\lambda, c) \in \mathscr{W}_0} D(\lambda, c) \tag{5.9}$$

for arbitrary subsets $\mathscr{W}_0$ of $\mathscr{W}$.

LEMMA 5.1. $\mathscr{C}^{**} \subseteq \mathscr{C}^{*}$.

PROOF. First we show $\mathscr{C}^{**} \subseteq \mathscr{C}$. Take $(\lambda, c) \in \mathscr{W}$ and let $\{\theta^{(i)}\}$ be the corresponding sequence given in (5.6). Since $\overline{K}(\Theta'(N))$ is a cone,

$$\theta^{(i)}/\|\theta^{(i)}\| \in \overline{K}(\Theta'(N)) \quad \text{for } \|\theta^{(i)}\| > N.$$

Thus by (5.8), $\lambda \in K_1$ of (5.3), hence by (5.2), $D(\lambda, c) \in \mathscr{C}$. Since $\mathscr{C}$ is closed under intersections, any C in (5.9) is contained in $\mathscr{C}$, so that $\mathscr{C}^{**} \subseteq \mathscr{C}$.

Next we show that $D(\lambda, c)$ satisfies (3.2). The lemma will follow since $\mathscr{C}^{*}$ is closed under intersections. We use Stein's (1956) proof. As in (3.2), take $\psi \in \Phi(D(\lambda, c))$ and suppose $r_\theta(\psi') \le r_\theta(\psi)$ for all $\theta \in \Theta$. Using (1.5), we can write

$$\begin{aligned} 0 &\ge a(\theta^{(i)})\exp\{c\|\theta^{(i)}\|\}\left(r_{\theta^{(i)}}(\psi') - r_{\theta^{(i)}}(\psi)\right) \\ &= \int_{D(\lambda, c)} I_i(x)\nu(dx) + \int_{D(\lambda, c)^c} I_i(x)\nu(dx), \end{aligned} \tag{5.10}$$

where

$$I_i(x) = (\psi(x) - \psi'(x))\exp\{\|\theta^{(i)}\|(\lambda^t x - c)\}\exp\left\{\left(\theta^{(i)} - \|\theta^{(i)}\|\lambda\right)^t x\right\}. \tag{5.11}$$

By (5.6)(ii) and (5.7)

$$|I_i(x)| \le \exp\{\tau\|x\|\} \quad \text{for } x \in D(\lambda, c). \tag{5.12}$$

Thus by (5.6)(iii), the first integral on the right-hand side of (5.6) is bounded in i. Since $\psi(x) = 1$ a.e. on $D(\lambda, c)^c$, $I_i(x) \ge 0$ a.e. on $D(\lambda, c)^c$. Thus we can use Fatou's lemma on the second integral. Unless $\psi'(x) = 1$ a.e. on $D(\lambda, c)^c$, too, the limit infimum of that final integral will be $+\infty$, violating (5.10). Thus $\psi' \in \Phi(D(\lambda, c))$, proving (3.2) for $D(\lambda, c)$. □

We remark that $\mathscr{C}^{**}$ may be a proper subset of $\mathscr{C}$. For example, take $\Theta = \{(\theta_1, \theta_2)^t | \theta_1 \in \mathbb{R}, \theta_2 = \theta_1^2\}$. Then $K_1 = \{(0, \theta)^t | \theta \ge 0\}$, so that $\mathscr{C}$ consists of all sets of the form $\{x | x_2 \le c\}$ for $c \ge 0$. However

$$\|\theta - \|\theta\|(0,1)^t\| \ge \theta_1^2 \to \infty \quad \text{as } |\theta_i| \to be,$$

hence no vector λ can satisfy (5.6)(i) and (ii). Thus $\mathscr{W}$ is empty, so that $\mathscr{C}^{**}$ is empty. Note that we have not said whether $\mathscr{C}^{*}$ is empty.

6. Examples.

EXAMPLE 6.1 (Correlation with known variances). Suppose $(Y_1, \dots, Y_n)$ is a sample of independent bivariate normal random variables with means 0, variances 1 and unknown correlation ρ. We wish to test

$$H_0\colon \rho = 0 \quad \text{versus} \quad H_A\colon \rho \in (-1,1) - \{0\}. \tag{6.1}$$

We have here a two-dimensional curved exponential family. After reducing by sufficiency to $x = (x_1, x_2)^t$, where

$$x_1 = \tfrac{1}{2} \sum_{i=1}^{n} \left(y_{i1}^2 + y_{i2}^2 \right) \quad \text{and} \quad x_2 = \sum_{i=1}^{n} y_{i1} y_{i2}, \tag{6.2}$$

and by taking $a(\rho) = (1 - \rho^2)^{n/2}$, we can write

$$R_\theta(x) = \exp\{\theta^t x\}, \tag{6.3}$$

where

$$\theta_1 = \frac{-\rho^2}{1 - \rho^2} \quad \text{and} \quad \theta_2 = \frac{\rho}{1 - \rho^2}. \tag{6.4}$$

Here

$$\Theta = \left\{ \left(\frac{-\rho^2}{1 - \rho^2}, \frac{\rho}{1 - \rho^2} \right) | \rho \in (-1,1) \right\} \tag{6.5}$$

is a one-dimensional curve in $\mathbb{R}^2$ as pictured in Figure 1.

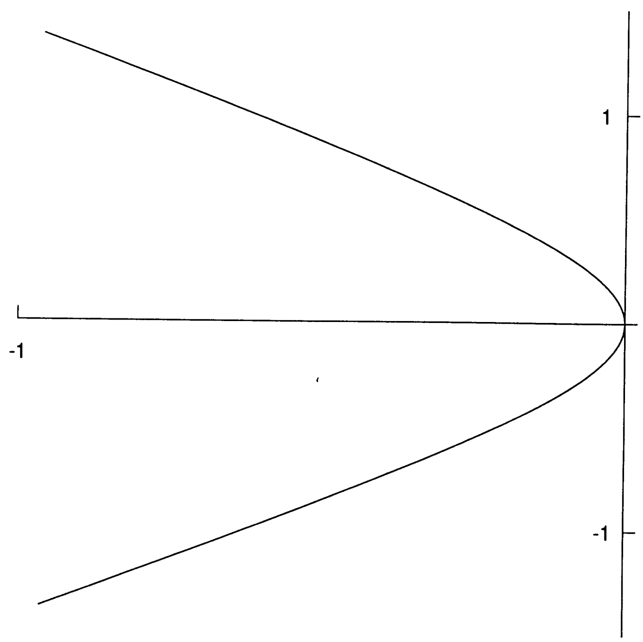

FIG. 1. *The parameter space* Θ.

To find $\Lambda(H)$, note that as $\theta_2 \to 0$,

$$\theta_1 = -\theta_2^2 + o(\theta_2^2). \tag{6.6}$$

Thus ignoring $o(\theta_2^2)$ terms, we have that, locally,

$$(\theta, \theta\theta^t) \doteq \left(\begin{pmatrix} -\theta_2^2 \\ \theta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \theta_2^2 \end{pmatrix} \right). \tag{6.7}$$

It can then be shown that

$$\lambda_{10} \equiv \lambda_1 - \int_{\overline{\Theta}(\alpha) - \{0\}} \frac{\theta_1}{\|\theta\|^2} H(d\theta) \tag{6.8}$$

is finite, and

$$\Lambda(H) = \left\{ (\lambda, M) | \lambda_{10} = -\overline{M}_0, \lambda_2 \in \mathbb{R}, M = \begin{pmatrix} 0 & 0 \\ 0 & \overline{M}_0 \end{pmatrix} \text{ and } \overline{M}_0 \ge 0 \right\}. \tag{6.9}$$

From (6.9) it follows that Δ_0 is not a product space, so that the inclusion in (4.14) is strict in this case. The local terms in $d(x)$ are now quadratics restricted to be of the form

$$\lambda^t l(x) + \frac{1}{2}\langle M_0, V(x) \rangle = \lambda_2 x_2 + \frac{\overline{M}_0}{2}(x_2^2 - 2x_1) + x_1 \int_{\overline{\Theta}(\alpha) - \{0\}} \frac{\theta_1}{\|\theta\|^2} H(d\theta)$$

for $(\lambda, M) \in \Lambda(H)$.

To obtain $\mathscr{C}$, it is straightforward to show that from (5.3), $K_1 \cap \{\theta | \|\theta\| = 1\} = \{(1/\sqrt{2})(-1, 1)^t, (1/\sqrt{2})(-1, -1)^t\}$. Thus by (5.2), $\mathscr{C}$ contains all sets of the form

$$C = \{x | x_1 - x_2 \ge c_1 \text{ and } x_1 + x_2 \ge c_2\} \tag{6.10}$$

for $c_1, c_2 \in [0, \infty]$. It is also easy to show that

$$\mathscr{W} = \left\{ \frac{1}{\sqrt{2}}(-1, 1)^t, \frac{1}{\sqrt{2}}(-1, -1)^t \right\} \times [-\infty, \infty].$$

Thus by Lemma 5.1, all $C \in \mathscr{C}$ satisfy (3.2). It can also be shown that Assumption 3.1(ii)–(iv) hold for all tests $\phi \in \Phi$, hence Φ is minimal complete. See Marden (1981) for details in the one-sided case.

Example 6.2 (Location parameter in the double exponential case). Suppose $(X_1, \dots, X_n)$ is a sample of independent observations from the double exponential distribution with location parameter θ, so that the density X_i with respect to Lebesgue measure on $\mathbb{R}$ is

$$g_\theta(x_i) = \tfrac{1}{2} \exp\{-|x_i - \theta|\}. \tag{6.11}$$

We test (1.1) with $\Theta = \mathbb{R}$. Since $g_\theta(x_i)$ is not differentiable with respect to θ at $\theta = x_i$, we need to delete $x_i = 0$ from the sample space in order to satisfy the

Local Assumption 2.2. Thus let $\mathscr{X} = [\mathbb{R} - \{0\}]^n$. Now (1.2) is

$$f_\theta(x) \equiv \exp\Big\{-\sum\big[|x_i - \theta| - |x_i|\big]\Big\}. \tag{6.12}$$

When $|\theta| < \min\{|x_i|, \dots, |x_n|\}$, $f_\theta(x) = \exp(\theta T)$, where

$$T = \sum_{i=1}^{n} \operatorname{sgn}(x_i) \quad \text{and} \quad \operatorname{sgn}(x_i) = \begin{cases} -1, & \text{if } x_i < 0, \\ 1, & \text{if } x_i > 0. \end{cases} \tag{6.13}$$

The statistic T is the one used in the sign test. When

$$|\theta| > \max\{|x_i|, \dots, |x_n|\}, \qquad f_\theta(x) = \exp\Big\{-n|\theta| + \sum\big[|x_i| + (\operatorname{sgn}\theta)x_i\big]\Big\}.$$

Thus if we choose $a(\theta) = \exp\{b(\theta)\}$, where

$$b(\theta) = \begin{cases} 0, & \text{if } |\theta| \le 1, \\ 2n(|\theta| - 1), & \text{if } 1 \le |\theta| < 2, \\ n|\theta|, & \text{if } 2 \le |\theta|, \end{cases}$$

we have from (2.1) that

$$R_\theta(x) = \begin{cases} \exp\{\theta T\}, & \text{if } |\theta| \le \min\{|x_1|, \dots, |x_n|, 1\}, \\ \exp\Big\{\sum|x_i| + \operatorname{sgn}\theta \sum x_i\Big\}, & \text{if } |\theta| \ge \max\{|x_1|, \dots, |x_n|, 2\} \end{cases} \tag{6.14}$$

and is given by $\exp\{b(\theta)\}f_\theta(x)$ in any case. Thus Assumptions 2.1 and 2.2 are seen to hold, with

$$l(x) = T \quad \text{and} \quad V(x) = T^2. \tag{6.15}$$

Consider the Asymptotic Assumption 2.3. Note that for any x, $R_\theta(x)$ is bounded away from 0 and ∞ for all θ. Thus (2.11) holds for all x or no x, i.e., $\mathscr{C} = \{\varnothing, \mathscr{X}\}$. If $C = \mathscr{X}$, then $\limsup_{i\to\infty} G_i(\Theta'(\alpha)) < \infty$, hence there is a limit G as in (2.9) for $G \in \mathscr{F}(\overline{\Theta}'(\alpha))$ and (2.10) holds since $R_\theta(x)$ is bounded. It need not be that (2.54) holds with $G^* = G$, hence we cannot always write d as in (2.55).

EXAMPLE 6.3 (Restricted normal means). Suppose Z_1 and Z_2 are independent, $Z_1 \sim N(\mu_1, \sigma_1^2)$, $Z_2 \sim N(\mu_2, \sigma_2^2)$, with σ_1^2 and σ_2^2 known. We have an exponential family (1.3), where we take

$$\theta = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \qquad x = \begin{pmatrix} z_1/\sigma_1^2 \\ z_2/\sigma_2^2 \end{pmatrix}. \tag{6.16}$$

We suppose $|\mu_1| \ge |\mu_2|$, so that

$$\Theta = \left\{ \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \middle| |\mu_1| \ge |\mu_2| \right\}. \tag{6.17}$$

It is straightforward to verify Assumptions 2.1–2.3. Take $\phi \in \Phi$. Assumption 3.1(i) holds since Θ is a closed cone; (iii) follows since each C is convex and ν is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^2$. To prove (iv), look at d in (5.5). If $(M_0^*, H^{**}) \neq (0,0)$, then d is *strictly* convex in at least one x_i. If $(M_0^*, H^{**}) = (0,0)$ but $\lambda^{**} \neq 0$, then d is either strictly increasing or strictly decreasing in at least one x_i. Thus if $(\lambda^*, M_0^*, H^{**}) \neq (0,0,0)$, (iv) holds. If $(\lambda^*, M_0^*, H^{**}) = (0,0,0)$, then $(\lambda, M_0, H, G) = (0,0,0,0)$ in (2.2), so that $d(x) \equiv -c$ and $c \neq 0$. Hence the set in (iv) is empty.

To prove (ii), take $G_i = GI_{\Theta'(i)}$ in Lemma 2.5, so that (3.4) holds by the Monotone Convergence Theorem and the remark following the lemma. Also, for any $\alpha > 0$, we can find a constant $K < \infty$ such that

$$\sup_{\theta \in \Theta(\alpha)} \left| \frac{R_\theta^{(2)}(x)}{\|\theta\|^2} \right| \leq K\left(\frac{z_1^2}{\sigma_1^4} + \frac{z_2^2}{\sigma_2^4} \right). \tag{6.18}$$

Since the upper bound in (6.18) is integrable with respect to ν, we can use the Dominated Convergence Theorem together with Monotone Convergence Theorem, to show that the limit and integral can be interchanged in (3.5).

Thus by Lemma 3.2 Φ is minimal complete. We now look at some specific tests.

First consider the test with acceptance region

$$\{x | z_1^2 + z_2^2 \leq \delta^2\} \equiv \{x | \sigma_1^4 x_1^2 + \sigma_2^4 x_2^2 \leq \delta^2\}, \quad \delta > 0. \tag{6.19}$$

We will show that it is admissible if and only if $\sigma_1^2 \geq \sigma_2^2$. Since $|\mu_1| \geq |\mu_2|$ in Θ, we can show that

$$\left(0, \begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix}\right) \in \Delta_0 \tag{6.20}$$

for any $m_{11} \geq m_{22} \geq 0$. Thus by taking in (2.24) $C = \mathscr{X}$, $H = 0$, $G = 0$, and (λ_0, M_0) as in (6.18), we have that any test with acceptance region

$$\{x | m_{11} x_1^2 + m_{22} x_2^2 \leq \delta\}, \qquad m_{11} \geq m_{22} \geq 0 \tag{6.21}$$

is admissible. Thus if $\sigma_1^2 \geq \sigma_2^2$, (6.18) is (6.20) with $(m_{11}, m_{22}) = (\sigma_1^4, \sigma_2^4)$. Now if $\sigma_1^2 < \sigma_2^2$, (6.20) cannot be written as (6.18). In fact, it cannot be written as in (2.24), proving it inadmissible.

To prove this fact, we first note that the problem and test (6.18) are invariant under the group G of sign changes: $g \circ (x_1, x_2) \to (e_1 x_1, e_2 x_2)$, where $e_1, e_2 \in \{\pm 1\}$. Thus we can invoke Remark 2.7. It is easy to show that

$$\bar{l}(x) = 0, \qquad \bar{V}(x) = \begin{pmatrix} x_1^2 & 0 \\ 0 & x_2^2 \end{pmatrix} \tag{6.22}$$

and

$$\bar{R}_\theta(x) = \tfrac{1}{4}\left(e^{|\mu_1|x_1} + e^{-|\mu_1|x_1}\right)\left(e^{|\mu_2|x_2} + e^{-|\mu_2|x_2}\right). \tag{6.23}$$

Now suppose (6.19) is the test (2.24) with the quantities barred as in (6.21) and (6.22). It is clear that $0 \in C$, hence we can write

$$d(x) = \bar{m}_{11} x_1^2 + \bar{m}_{22} x_2^2 + \int_{\Theta - \{0\}} \frac{\bar{R}_\theta(x) - 1}{\mu_1^2 + \mu_2^2} \bar{H}(d\mu) - c, \tag{6.24}$$

where $\bar{m}_{11} \geq \bar{m}_{22} \geq 0$ and $\bar{H} \in \mathscr{F}(\theta - \{0\})$ is G invariant. By the nature of the set C in (5.2), where $K_1 = \Theta$, and since $d(x)$ is continuous, we must have that

(6.25) $$d(x) = 0 \quad \text{if } x \in \{x | \sigma_1^4 x_1^2 + \sigma_2^4 x_2^2 = \delta^2\} \cap \{x | \, |x_1| \le |x_2|\} \equiv A.$$

Consider the points $x, y \in A$, defined by

$$x = \begin{pmatrix} 0 \\ \dfrac{\delta}{\sigma_2^2} \end{pmatrix}, \qquad y = \frac{\delta}{\sqrt{\sigma_1^4 + \sigma_2^4}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since $\sigma_1^2 < \sigma_2^2$, $\overline{m}_{22} x_2^2 < \overline{m}_{11} y_1^2 + \overline{m}_{22} y_2^2$ if $(\overline{m}_{11}, \overline{m}_{22}) \neq (0,0)$ and it can be shown that

(6.26) $$\overline{R}_\theta(x) < \overline{R}_\theta(y) \quad \text{if } \theta \neq 0.$$

Thus unless $(\overline{m}_{11}, \overline{m}_{22}, \overline{H}) = (0, 0, 0)$, $d(x) < d(y)$, violating (6.24). If $(\overline{m}_{11}, \overline{m}_{22}, \overline{H}) = (0,0,0)$, then $d(x) \equiv 0$, and (6.24) cannot hold. Thus (6.19) is inadmissible.

We note that the test with acceptance region

$$\left\{ x \Big| \frac{z_1^2}{\sigma_1^2} + \frac{z_2^2}{\sigma_2^2} \le \delta^2 \right\} \equiv \{x | \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 \le \delta^2\}$$

is also admissible if and only if $\sigma_1^2 \ge \sigma_2^2$.

Now consider the Likelihood Ratio Test (L.R.T.). Its acceptance region is given in Figure 2.

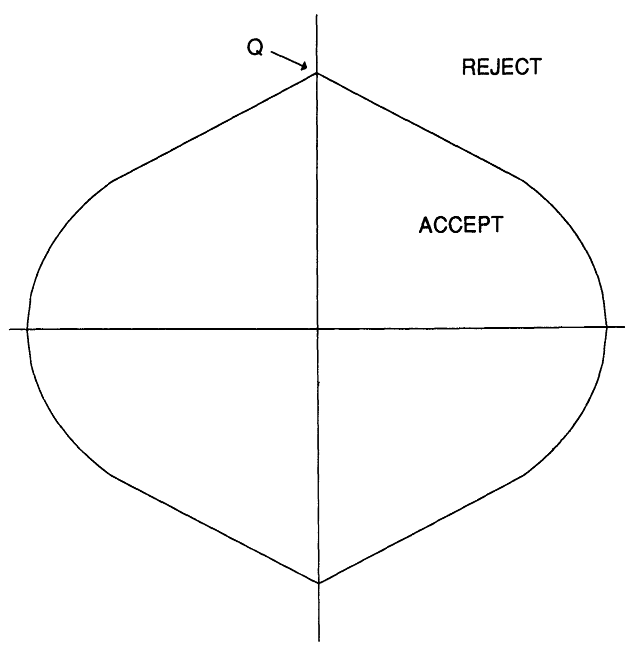

FIG. 2. *The acceptance region of the L.R.T. when* $\sigma_1^2 = 2$ *and* $\sigma_2^2 = 1$.

As in Birnbaum (1955), it can be shown to be admissible since it is $1 - I_C$, where for some κ,

$$C = \bigcap_{\theta \in \Theta} \left\{ x | \mu_1 x_1 + \mu_2 x_2 \leq \frac{1}{2} \left(\frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} \right) + \kappa \right\}.$$

It is easily seen from (5.2) that $C \in \mathscr{C}$.

Now suppose we bound the parameter space, that is, consider the parameter space,

$$\Theta_B = \{\theta \in \Theta | \, \|\theta\| \leq B\}$$

for some very large B. Then the L.R.T. will still be as above, but the set $\mathscr{C}$ will consist of only $\mathscr{X}$ and $\varnothing$ since K_1 in (5.3) is empty. To be admissible, the L.R.T. would have to have acceptance region $\{x | d(x) \leq c\}$. But d is analytic in each x_i, and Brown (1986) shows that if d is analytic, there cannot be a sharp point like Q. Thus the L.R.T. is inadmissible.

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